1. Let $f(\cdot)$ be a non-negative measurable function on a σ-finite measure space $(\Omega, \mathcal{B}, \mu)$. Show that $f(\cdot) = 0$, μ-a.e. iff $\int_\Omega f(\omega) d\mu(\omega) = 0$.

   Solution: See Proposition 2.3.4, Page 40, of 'Measure and Probability' by S R Athreya and V S Sunder (Universities Press).

   $\square$

2. Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space. Let $\mathcal{A} \subseteq \mathcal{B}$ be a sub-σ-algebra. Let X be real valued random variable having finite expectation. Show that there is an $\mathcal{A}$-measurable and integrable random variable Y, s.t. $\int_A X(\omega) d\mathbb{P}(\omega) = \int_A Y(\omega) d\mathbb{P}(\omega)$ for all $A \in \mathcal{A}$, and that Y is unique $\mathbb{P}$-a.e.

Solution: See 'Measure Theory an Probability Theory' (Springer Texts in Statistics) by K B Athreya and S N Lahiri, Page 385-386, Theorem 12.1.4, Remark 12.1.2, Remark 12.1.3.
Can also see, Theorem 3.4.1, Page 53, of 'Measure and Probability' by S R Athreya and V S Sunder (Universities Press).

$\square$

3. Let $\mathcal{B}(\mathbb{R})$, $\mathcal{B}(\mathbb{R}^2)$ denote respectively the Borel σ-algebras of $\mathbb{R}$, $\mathbb{R}^2$. Show that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$, where the r.h.s. is the smallest σ-algebra containing all measurable rectangles.

Solution: See Example 18.1, Page-231, of 'Probability and Measure' by P Billingsley (third edition, Wiley).

$\square$

4. Let X, X_n, $n = 1, 2, \cdots$ be real valued random variables defined on a probability space $(\Omega, \mathcal{B}, P)$.

(i) If $X_n \to X$ in probability, show that there is a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ s.t. $X_{n_k} \to X$ P-a.e. as $k \to \infty$.
(ii) Let $X_n \to X$ in probability. Suppose there is a random variable Y which is integrable w.r.t. P, s.t. $|X_n(\omega)| \leq Y(\omega),\ \omega \in \Omega,\ n \geq 1$. Show that X is integrable w.r.t. P, and that

$$\int_\Omega X(\omega) dP(\omega) = \lim_{n\to\infty} \int_\Omega X_n(\omega) dP(\omega).$$

(iii) If $X_n \to b$ as $n \to \infty$, where $b \in \mathbb{R}$ is a constant, show that $X_n \to b$ in probability.

Solution: (i) See Lemma 5.2.4, Page 81, of 'Measure and Probability' by S R Athreya and V S Sunder (Universities Press).
(ii) Convergence in probability implies convergence in distribution. Then by applying Dominated Convergence Theorem, we obtain the result.

(iii) Pointwise convergence implies convergence almost every where and that implies convergence in probability, hence the result.

$\square$

5.(i) Let X, Y be independent real valued random variables. Express the characteristic function of $X+Y$ in terms of the characteristic functions of X and Y.
(ii) For $n = 1, 2, \cdots$ let $Z_n = X + Y_n$, where X and Y_n are independent real valued random variables, and Y_n has $N(0, \frac{1}{n})$ distribution. Does $\{Z_n\}$ converge in distrubution? Justify.

Solution: (i)

$$\phi_X(t) = E(e^{itX}),$$
$$\phi_Y(t) = E(e^{itY}).$$

Now

$$\begin{aligned}\phi_{X+Y}(t) &= E(e^{it(X+Y)}) \\ &= E(e^{itX}) \cdot E(e^{itY}) \ \ \text{[as } X,\ Y \text{ independent r.v.]} \\ &= \phi_X(t) \cdot \phi_Y(t).\end{aligned}$$

(ii)

$$\phi_{Z_n}(t) = \phi_{X+Y_N}(t) = \phi_X(t) \cdot \phi_{Y_N}(t).$$

As, $Y_N \sim N(0, \frac{1}{N})$, therefore $\phi_{Y_N}(t) = e^{-\frac{t^2}{2n^2}}$.

$$\lim_{n\to\infty} \phi_{Y_n}(t) = \lim_{n\to\infty} e^{-\frac{t^2}{2n^2}} = 1.$$

Hence,

$$\lim_{n\to\infty} \phi_{Z_n}(t) = \phi_X(t) \lim_{n\to\infty} \phi_{Y_n}(t) = \phi_X(t).$$

Therefore, we conclude that, $\{Z_n\}$ converge in distrubution. $\square$